

# NUT-Charged Black Holes in Gauss-Bonnet Gravity

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## Abstract

We investigate the existence of Taub-NUT/bolt solutions in Gauss-Bonnet gravity and obtain the general form of these solutions in  $d$  dimensions. We find that for all non-extremal NUT solutions of Einstein gravity having no curvature singularity at  $r = N$ , there exist NUT solutions in Gauss-Bonnet gravity that contain these solutions in the limit that the Gauss-Bonnet parameter  $\alpha$  goes to zero. Furthermore there are no NUT solutions in Gauss-Bonnet gravity that yield non-extremal NUT solutions to Einstein gravity having a curvature singularity at  $r = N$  in the limit  $\alpha \rightarrow 0$ . Indeed, we have non-extreme NUT solutions in  $2 + 2k$  dimensions with non-trivial fibration only when the  $2k$ -dimensional base space is chosen to be  $\mathbb{CP}^{2k}$ . We also find that the Gauss-Bonnet gravity has extremal NUT solutions whenever the base space is a product of 2-torii with at most a 2-dimensional factor space of positive curvature. Indeed, when the base space has at most one positively curved two dimensional space as one of its factor spaces, then Gauss-Bonnet gravity admits extreme NUT solutions, even though there a curvature singularity exists at  $r = N$ . We also find that one can have bolt solutions in Gauss-Bonnet gravity with any base space with factor spaces of zero or positive constant curvature. The only case for which one does not have bolt solutions is in the absence of a cosmological term with zero curvature base space.

# 1 Introduction

Four-dimensional Taub-NUT and Taub-bolt-AdS solutions of Einstein gravity play a central role in the construction of diverse and interesting M-theory configurations. Indeed, the 4-dimensional Taub-NUT-AdS solution provided the first test for the AdS/CFT correspondence in spacetimes that are only locally asymptotically AdS [1, 2, 3], and the Taub-NUT metric is central to the supergravity realization of the  $D6$  brane of type IIA string theory [4]. It is therefore natural to suppose that the generalization of these solutions to the case of Lovelock gravity, which is the low energy limit of supergravity, might provide us with a window on some interesting new corners of M-theory moduli space.

The original four-dimensional solution [5, 6] is only locally asymptotic flat. The spacetime has as a boundary at infinity a twisted  $S^1$  bundle over  $S^2$ , instead of simply being  $S^1 \times S^2$ . There are known extensions of the Taub-NUT solutions to the case when a cosmological constant is present. In this case the asymptotic structure is only locally de Sitter (for positive cosmological constant) or anti-de Sitter (for negative cosmological constant) and the solutions are referred to as Taub-NUT-(A)dS metrics. In general, the Killing vector that corresponds to the coordinate that parameterizes the fibre  $S^1$  can have a zero-dimensional fixed point set (called a NUT solution) or a two-dimensional fixed point set (referred to as a ‘bolt’ solution). Generalizations to higher dimensions follow closely the four-dimensional case [7, 8, 9, 10, 11, 12, 13, 14].

In this paper we consider Taub-NUT metrics in second order Lovelock gravity (referred to as Gauss-Bonnet gravity), which is a higher-dimensional generalization of Einstein gravity. In higher dimensions it is possible to use other consistent theories of gravity with actions more general than that of the Einstein-Hilbert action. Such an action may be written, for example, through the use of string theory. The effect of string theory on classical gravitational physics is usually that of modifying the low energy effective action that describes gravity at the classical level [15]. This effective action consists of the Einstein-Hilbert action plus curvature-squared terms and higher powers as well, and in general gives rise to fourth-order field equations containing ghosts. However if the effective action contains the higher powers of curvature in particular combinations, then only second-order field equations are produced and consequently no ghosts arise [16]. The effective action obtained by this argument is precisely of the form proposed by Lovelock [17].

Until now whether or not higher derivative gravity admits solutions of the Taub-NUT/bolt form has been an open question. Due to the nonlinearity of the field equations, it is very difficult to find nontrivial exact analytic solutions of Einstein’s equations modified with higher curvature terms. In most cases, one has to adopt some approximation methods or find solutions numerically. However, exact static spherically symmetric black hole solutions of second and third order Lovelock gravity have been obtained in Refs. [18, 19, 20], and of Einstein-Maxwell-Gauss-Bonnet model in Ref. [21]. Black hole solutions with nontrivial topology and their thermodynamics in this theory have been also studied [22]. All of these solutions in Gauss-Bonnet gravity are static. Recently two new classes of rotating solutions of second order Lovelock gravity have been introduced and their thermodynamics have been investigated [23].

Here we investigate the existence of Taub-NUT and Taub-bolt solutions of Gauss-Bonnet

gravity. We find that exact solutions exist, but that Lovelock gravity introduces some features not present in higher-dimensional Einstein-Hilbert gravity. The form of the metric function is sensitive to the base space over which the circle is fibred. Furthermore, we find that pure non-extreme NUT solutions only exist if the base space has a single factor of maximal dimensionality. We conjecture that this is a general property of Gauss-Bonnet gravity in any dimension.

The outline of our paper is as follows. We give a brief review of the field equations of second order Lovelock gravity in Sec. 2. In Sec. 3, we obtain all possible Taub-NUT/bolt solutions of Gauss-Bonnet gravity in six dimensions. The structure of these solutions suggests two conjectures, which we posit and then check in the remainder of the paper. Then, in sections 4 and 5, we present all kind Taub-NUT/bolt solutions of Gauss-Bonnet gravity in eight and ten dimensions and check the conjectures for them. In Sec 6, we extend our study to the  $d$ -dimensional case, and find the general expressions for the Taub-NUT/bolt solutions. We finish our paper with some concluding remarks.

## 2 Field Equations

The most fundamental assumption in standard general relativity is the requirement that the field equations be generally covariant and contain at most second order derivatives of the metric. Based on this principle, the most general classical theory of gravitation in  $d$  dimensions is Lovelock gravity. The gravitational action of this theory can be written as

$$I_G = \int d^d x \sqrt{-g} \sum_{k=0}^{[d/2]} \alpha_k \mathcal{L}_k \quad (1)$$

where  $[z]$  denotes the integer part of  $z$ ,  $\alpha_k$  is an arbitrary constant and  $\mathcal{L}_k$  is the Euler density of a  $2k$ -dimensional manifold,

$$\mathcal{L}_k = \frac{1}{2^k} \delta_{\rho_1 \sigma_1 \dots \rho_k \sigma_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k} R_{\mu_1 \nu_1}^{\rho_1 \sigma_1} \dots R_{\mu_k \nu_k}^{\rho_k \sigma_k} \quad (2)$$

In Eq. (2)  $\delta_{\rho_1 \sigma_1 \dots \rho_k \sigma_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k}$  is the generalized totally anti-symmetric Kronecker delta and  $R_{\mu\nu}^{\rho\sigma}$  is the Riemann tensor. We note that in  $d$  dimensions, all terms for which  $k > [d/2]$  are identically equal to zero, and the term  $k = d/2$  is a topological term. Consequently only terms for which  $k < d/2$  contribute to the field equations. Here we restrict ourselves to the first three terms of Lovelock gravity, which is known as Gauss-Bonnet gravity. In this case the action is

$$I_G = \frac{1}{2} \int_{\mathcal{M}} dx^d \sqrt{-g} [-2\Lambda + R + \alpha(R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4R_{\mu\nu} R^{\mu\nu} + R^2)] \quad (3)$$

where  $\Lambda$  is the cosmological constant,  $R$ ,  $R_{\mu\nu\rho\sigma}$ , and  $R_{\mu\nu}$  are the Ricci scalar and Riemann and Ricci tensors of the spacetime, and  $\alpha$  is the Gauss-Bonnet coefficient with dimension  $(\text{length})^2$ . Since it is positive in heterotic string theory [18] we shall restrict ourselves to the case  $\alpha > 0$ . The first term is the cosmological term, the second term is just the Einstein term,

and the third term is the second order Lovelock (Gauss-Bonnet) term. From a geometric point of view, the combination of these terms in five-dimensional spacetimes is the most general Lagrangian producing second order field equations, analogous to the situation in four-dimensional gravity for which the Einstein-Hilbert action is the most general Lagrangian producing second order field equations.

Varying the action with respect to the metric tensor  $g_{\mu\nu}$ , the vacuum field equations are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} - \alpha \left\{ \frac{1}{2}g_{\mu\nu}(R_{\kappa\lambda\rho\sigma}R^{\kappa\lambda\rho\sigma} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2) \right. \\ \left. - 2RR_{\mu\nu} + 4R_{\mu\lambda}R^\lambda_\nu + 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2R_\mu^{\rho\sigma}R_{\nu\rho\sigma\lambda} \right\} = 0 \end{aligned} \quad (4)$$

Equation (4) does not contain derivatives of the curvature, and therefore derivatives of the metric higher than two do not appear.

We seek Taub-NUT solutions of the field equations (4). In constructing these metrics the idea is to regard the Taub-NUT space-time as a  $U(1)$  fibration over a  $2k$ -dimensional base space endowed with an Einstein-Kähler metric  $g_{\mathcal{B}}$ . Then the Euclidean section of the  $(2k+2)$ -dimensional Taub-NUT spacetime can be written as:

$$ds^2 = F(r)(d\tau + NA)^2 + F^{-1}(r)dr^2 + (r^2 - N^2)g_{\mathcal{B}} \quad (5)$$

where  $\tau$  is the coordinate on the fibre  $S^1$  and  $A$  has a curvature  $F = dA$ , which is proportional to some covariantly constant 2-form. Here  $N$  is the NUT charge and  $F(r)$  is a function of  $r$ . The solution will describe a ‘NUT’ if the fixed point set of the  $U(1)$  isometry  $\partial/\partial\tau$  (i.e. the points where  $F(r) = 0$ ) is less than  $2k$ -dimensional and a ‘bolt’ if the fixed point set is  $2k$ -dimensional.

### 3 Six-dimensional Solutions

In this section we study the six-dimensional Taub-NUT/bolt solutions (5) of Gauss-Bonnet gravity. We find that the function  $F(r)$  for all the possible choices of the base space  $\mathcal{B}$  can be written in the form

$$\begin{aligned} F(r) &= \frac{(r^2 - N^2)^2}{12\alpha(r^2 + N^2)} \left( 1 + \frac{p\alpha}{(r^2 - N^2)} - \sqrt{B(r) + C(r)} \right) \\ B(r) &= 1 + \frac{4p\alpha N^2(r^4 + 6r^2N^2 + N^4) + 12\alpha mr(r^2 + N^2)}{(r^2 - N^2)^4} \\ &\quad + \frac{12\alpha\Lambda(r^2 + N^2)}{5(r^2 - N^2)^4} (r^6 - 5N^2r^4 + 15N^4r^2 + 5N^6) \end{aligned} \quad (6)$$

where  $p$  is the sum of the dimensions of the curved factor spaces of  $\mathcal{B}$ , and the function  $C(r)$  depends on the choice of the base space  $\mathcal{B}$ .

We first study the solutions for which all factor spaces of  $\mathcal{B}$  are curved. The first possibility is that the base space is  $\mathcal{B} = \mathbb{CP}^2$ , where  $A$  and the  $\mathbb{CP}^2$  metric are:

$$A_2 = 6\sin^2\xi_2(d\psi_2 + \sin^2\xi_1d\psi_1) \quad (7)$$

$$\begin{aligned} d\Sigma_2^2 &= 6\{d\xi_2^2 + \sin^2\xi_2\cos^2\xi_2(d\psi_2 + \sin^2\xi_1d\psi_1)^2 \\ &\quad + \sin^2\xi_2(d\xi_1^2 + \sin^2\xi_1\cos^2\xi_1d\psi_1^2)\} \end{aligned} \quad (8)$$

The function  $C(r)$  for  $\mathcal{B} = \mathbb{CP}^2$  is:

$$C_{\mathbb{CP}^2} = -\frac{16\alpha^2(r^4 + 6r^2N^2 + N^4)}{(r^2 - N^2)^4} \quad (9)$$

The second possibility is  $\mathcal{B} = S^2 \times S^2$ , where  $S^2$  is the 2-sphere with  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ , and the one-form  $A$  is

$$A = 2\cos\theta_1 d\phi_1 + 2\cos\theta_2 d\phi_2 \quad (10)$$

In this case  $C(r)$  is

$$C_{S^2 \times S^2} = -\frac{32\alpha^2(r^4 + 4r^2N^2 + N^4)}{(r^2 - N^2)^4} \quad (11)$$

Note that the asymptotic behavior of these solutions for positive  $\alpha$  is locally flat when  $\Lambda$  vanishes, locally dS for  $\Lambda > 0$  and locally AdS for  $\Lambda < 0$  provided  $|\Lambda| < 5/(12\alpha)$ .

We note that the function  $F(r)$  given in (6) has the same form in the limit  $\alpha \rightarrow 0$ , and reduces to  $F(r)$  given in [12]. Factor spaces of zero curvature will be considered at the end of this section.

If the base space has all factor spaces curved, then the remaining possibilities are  $\mathcal{B} = S^2 \times H^2$  or  $\mathcal{B} = H^2 \times H^2$ , where  $H^2$  is the two-dimensional hyperboloid with metric  $d\Xi^2 = d\theta^2 + \sinh^2\theta d\phi^2$ . However we find that the function  $F(r)$  is not real for all values of  $r$  in the range  $0 \leq r \leq \infty$  for positive values of  $\alpha$ . The only case for which a black hole solution can be obtained is when  $\alpha$  and  $\Lambda$  are negative. As we mentioned earlier, we are interested in positive values of  $\alpha$ , and so we don't consider them here.

### 3.1 Taub-NUT Solutions

The solutions given in Eq. (6) describe NUT solutions, if (i)  $F(r = N) = 0$  and (ii)  $F'(r = N) = 1/(3N)$ . The first condition comes from the fact that all the extra dimensions should collapse to zero at the fixed point set of  $\partial/\partial\tau$ , and the second one ensures that there is no conical singularity with a smoothly closed fiber at  $r = N$ . Using these conditions, one finds that Gauss-Bonnet gravity in six dimensions admits NUT solutions with a  $\mathbb{CP}^2$  base space when the mass parameter is fixed to be

$$m_n = -\frac{16}{15}N(3\Lambda N^4 + 5N^2 - 5\alpha) \quad (12)$$

Computation of the Kretschmann scalar at  $r = N$  for the solutions given in the last section shows that the spacetime with  $\mathcal{B} = S^2 \times S^2$  has a curvature singularity at  $r = N$  in Einstein gravity, while the spacetime with  $\mathcal{B} = \mathbb{CP}^2$  has no curvature singularity at  $r = N$ . Thus, we conjecture that “*For all non-extremal NUT solutions of Einstein gravity having no curvature singularity at  $r = N$ , there exist NUT solutions in Gauss-Bonnet gravity that contain these solutions in the limit that the parameter  $\alpha$  vanishes. Furthermore there are no NUT solutions in Gauss-Bonnet gravity that yield non-extremal NUT solutions to Einstein gravity having a curvature singularity at  $r = N$  in the limit  $\alpha \rightarrow 0$ .*” Indeed, we have

non-extreme NUT solutions in  $2 + 2k$  dimensions with non-trivial fibration when the  $2k$ -dimensional base space is chosen to be  $\mathbb{CP}^2$ .

We will test this conjecture throughout the rest of the paper, and consider the case of extremal NUT solutions at the end of this section.

### 3.2 Taub-Bolt Solutions

The conditions for having a regular bolt solution are (i)  $F(r = r_b) = 0$  and (ii)  $F'(r_b) = 1/(3N)$  with  $r_b > N$ . Condition (ii), which again follows from the fact that we want to avoid a conical singularity at the bolt, together with the fact that the period of  $\tau$  will still be  $12\pi N$ , gives the following equation for  $r_b > N$  with the base space  $\mathbb{CP}^2$

$$3N\Lambda r_b^3 + (2 + 3\Lambda N^2)r_b^2 - N(4 + 3\Lambda N^2)r_b - 3\Lambda N^4 - 6N^2 + 8\alpha = 0 \quad (13)$$

which has at least one real solution. This real solution for  $N < \sqrt{\alpha}$  yields  $r_b > N$ , which is a bolt solution, while for  $N \geq \sqrt{\alpha}$ , there is no bolt solution – only the NUT solution can satisfy the regularity conditions.

As  $\Lambda$  goes to zero this real solution for  $r_b$  diverges. However setting  $\Lambda = 0$  we obtain the asymptotically locally flat case, and Eq. (13) becomes a quadratic equation with the following two roots:

$$r_b = N \pm 2\sqrt{(N^2 - \alpha)} \quad (14)$$

Note that we have only one possible bolt solution (provided  $N > \sqrt{\alpha}$ ), since the smaller root is less than  $N$ .

For the case of  $\mathcal{B} = S^2 \times S^2$ ,  $r_b$  can be found by solving the following equation:

$$3N\Lambda r_b^4 + 2r_b^3 - 6N(\Lambda N^2 + 1)r_b^2 - 2(N^2 - 4\alpha)r_b + 3\Lambda N^5 + 6N^3 - 12\alpha N = 0 \quad (15)$$

where the condition for having bolt solution(s) is that  $N \leq N_{\max}$ , where  $N_{\max}$  is the smaller root of the following equation:

$$\begin{aligned} & -144\Lambda^2(1 + 3\Lambda\alpha)(4 + 15\Lambda\alpha)N^{10} - 144\Lambda(72\Lambda^2\alpha^2 + 49\Lambda\alpha + 8)N^8 \\ & + 8(2916\Lambda^3\alpha^3 + 2331\Lambda^2\alpha^2 + 396\Lambda\alpha - 8)N^6 - 48\alpha(216\Lambda^2\alpha^2 + 51\Lambda\alpha - 4)N^4 \\ & + 3\alpha^2(1296\Lambda^2\alpha^2 + 360\Lambda\alpha - 55)N^2 + 64\alpha^3 = 0 \end{aligned} \quad (16)$$

Note that the above equation (16) reduces to the equation for  $N_{\max}$  of Einstein gravity when  $\alpha$  vanishes. For  $N < N_{\max}$  with negative  $\Lambda$ , we have two bolt solutions, while for  $N = N_{\max}$  we have only one bolt solution. It is straightforward to show that the value of  $N_{\max}$  is larger in Gauss-Bonnet gravity with respect to its value in Einstein gravity. As  $\Lambda \rightarrow 0$ , the larger bolt solution goes to infinity, so for the asymptotically flat solution, there is always one and only one bolt solution that is the real root of the following equation:

$$r_b^3 - 3Nr_b^2 - (N^2 - 4\alpha)r_b + 3N^3 - 6\alpha N = 0 \quad (17)$$

Eq. (17) has either one or three real roots larger than  $N$  since its left hand side is negative at  $r_b = N$ , and positive as  $r_b$  goes to infinity. However it cannot have three real roots larger

than  $N$ . This is because the product of the roots is equal to  $-3N(N^2 - 2\alpha) < 0$ , whereas the condition for having three real roots guarantees that  $N^2 > 2\alpha$ . Thus there exists one and only one real root larger than  $N$ , and so there is always one bolt solution. This kind of argument applies for all of the solutions we obtain in any dimension with any base space.

Finally, we note an additional regularity condition not present in Einstein gravity. The metric function  $F(r)$  is real for all values of  $r$  in the range  $0 \leq r \leq \infty$  provided  $\alpha < \alpha_{\max}$ , where  $\alpha_{\max}$  depends on the parameters of the metric. This is a general feature of solutions in Gauss-Bonnet gravity and also occurs for static solutions [18].

We also note that for  $\Lambda > 0$  the bolt radius increases with increasing  $\alpha$ , while for the case of  $\Lambda = 0$  it decreases as  $\alpha$  increases. For the case of  $\Lambda < 0$ , there are two bolt solutions provided  $N < N_{\max}$ . As  $\alpha$  increases the radius of smaller one decreases, while that of the larger solution increases. For  $N = N_{\max}$ , only the smaller bolt solution exists, and its radius  $r_b$  decreases as  $\alpha$  increases. These features happen for all the bolt solutions in any dimension.

### 3.3 Taub-NUT/Bolt Solutions with $T^2$ in the base

We now consider the Taub-NUT/bolt solutions of Gauss-Bonnet gravity when  $\mathcal{B}$  contains a 2-torus  $T^2$  with metric  $d\Gamma = d\eta^2 + d\zeta^2$ . There are two possibilities. The first possibility is to choose the base space  $\mathcal{B}$  to be  $S^2 \times T^2$  with 1-form

$$A = 2 \cos \theta d\phi + 2\eta d\zeta \quad (18)$$

and the second is  $T^2 \times T^2$  with 1-form

$$A = 2\eta_1 d\zeta_1 + 2\eta_2 d\zeta_2 \quad (19)$$

The function  $F(r)$  for the above two cases is given by Eq. (6), where  $C(r)$  is zero for  $\mathcal{B} = T^2 \times T^2$  and is

$$C_{ST} = \frac{4\alpha^2}{(r^2 - N^2)^2} \quad (20)$$

for  $\mathcal{B} = S^2 \times T^2$ . The topology  $\mathcal{B} = T^2 \times T^2$  at  $r \rightarrow \infty$  is  $\mathbb{R}^5$ . Although the boundary is topologically a direct product of the Euclidean time line and the spatial hypersurface  $(\eta_1, \eta_2, \zeta_1, \zeta_2)$ , the product is twisted and the boundary is not flat. Unlike the  $B = CP^2$  case, an immediate consequence of the  $T^2 \times T^2$  topology is that the Euclidean time period  $\beta = 4\pi/F'$  will not be fixed by the value of the nut parameter  $N$ .

Now we consider the NUT solutions. For these two cases the conditions of existence of NUT solutions are satisfied provided the mass parameter is

$$m_N = -\frac{16}{5}\Lambda N^5, \quad (21)$$

$$m_N = -\frac{8}{15}N^3(6\Lambda N^2 + 5) \quad (22)$$

for  $\mathcal{B} = T^2 \times T^2$  and  $\mathcal{B} = T^2 \times S^2$  respectively. Indeed for these two cases  $F'(r = N) = 0$ , and therefore the NUT solutions should be regarded as extremal solutions. Computing the Kretschmann scalar, we find that there is a curvature singularity at  $r = N$  for the spacetime

with  $\mathcal{B} = T^2 \times S^2$ , while the spacetime with  $\mathcal{B} = T^2 \times T^2$  has no curvature singularity at  $r = N$ .

This leads us to our second conjecture: “*Gauss-Bonnet gravity has extremal NUT solutions whenever the base space is a product of 2-torii with at most one 2-dimensional space of positive curvature*”. Indeed, when the base space has at most one two dimensional curved space as one of its factor spaces, then Gauss-Bonnet gravity admits an extreme NUT solution even though there exists a curvature singularity at  $r = N$ .

Next we consider the Taub-bolt solutions. Euclidean regularity at the bolt requires the period of  $\tau$  to be

$$\beta = \frac{8\pi r_b(r_b^2 - N^2 + 2\alpha)}{(r_b^2 - N^2)[1 - \Lambda(r_b^2 - N^2)]} \quad (23)$$

for  $\mathcal{B} = T^2 \times S^2$ , and

$$\beta = -\frac{8\pi r_b}{\Lambda(r_b^2 - N^2)} \quad (24)$$

for  $\mathcal{B} = T^2 \times T^2$ . As  $r_b$  varies from  $N$  to infinity, one covers the whole temperature range from 0 to  $\infty$ , and therefore we have non-extreme bolt solutions. Note that for the case of  $\Lambda = 0$ , there is no black hole solution with  $\mathcal{B} = T^2 \times T^2$ . This is also true for spherically symmetric solutions of Gauss-Bonnet gravity [22].

## 4 Eight-dimensional Solutions

In eight dimensions there are more possibilities for the base space  $\mathcal{B}$ . It can be a 6-dimensional space, a product of three 2-dimensional spaces, or the product of a 4-dimensional space with a 2-dimensional one. In all of these cases the form of the function  $F(r)$  is

$$\begin{aligned} F(r) &= \frac{(r^2 - N^2)^2}{8\alpha(5r^2 + 3N^2)} \left( 1 + \frac{4p\alpha}{3(r^2 - N^2)} - \sqrt{B(r) + C(r)} \right) \\ B(r) &= 1 - \frac{16\alpha m r (5r^2 + 3N^2)}{3(r^2 - N^2)^5} + \frac{16p\alpha N^2}{15(r^2 - N^2)^5} (r^6 - 15N^2 r^4 - 45N^4 r^2 - 5N^6) \\ &\quad + \frac{16\alpha\Lambda (5r^2 + 3N^2)}{105(r^2 - N^2)^5} (5r^8 - 28N^2 r^6 + 70N^4 r^4 - 140N^6 r^2 - 35N^8) \end{aligned} \quad (25)$$

where  $p$  is again the dimension of the curved factor spaces of  $\mathcal{B}$ , and the function  $C(r)$  depends on the choice of the base space.

The 1-form and the metric for the factor spaces  $S^2$ ,  $T^2$ , and  $\mathbb{CP}^2$  have been introduced in the last section. Here, for completeness we write down the metric and 1-form for the factor space  $\mathbb{CP}^3$ , and then we bring the function  $F(r)$  for the various base spaces in a table. The metric and the 1-form  $A$  for  $\mathbb{CP}^3$  may be written as:

$$d\Sigma_3^2 = 8\{d\xi_3^2 + \sin^2 \xi_3 \cos^2 \xi_3 (d\psi_3 + \frac{1}{6}A_2)^2 + \frac{1}{6} \sin^2 \xi_3 d\Sigma_2^2\} \quad (26)$$

$$A_3 = 8 \sin^2 \xi_3 \{d\psi_3 + \sin^2 \xi_2 (d\psi_2 + \sin^2 \xi_1 d\psi_1)\} \quad (27)$$

The function  $C(r)$  for various base spaces are:



$\mathcal{B}$	$(r^2 - N^2)^5 C(r) / \alpha^2$
$\mathbb{CP}^3$	$-16(r^6 - 15N^2r^4 - 45N^4r^2 - 5N^6)$
$T^2 \times T^2 \times T^2$	0
$T^2 \times T^2 \times S^2$	$\frac{64}{9}(r^2 - N^2)^3$
$T^2 \times S^2 \times S^2$	$-\frac{64}{9}(r^6 - 15N^2r^4 - 45N^4r^2 - 5N^6)$
$S^2 \times S^2 \times S^2$	$-\frac{128}{3}(r^6 - 9N^2r^4 - 21N^4r^2 - 3N^6)$
$T^2 \times \mathbb{CP}^2$	$\frac{128}{27}(r^6 + 9N^2r^4 + 51N^4r^2 + 3N^6)$
$S^2 \times \mathbb{CP}^2$	$-\frac{64}{27}(13r^6 - 135N^2r^4 - 345N^4r^2 - 45N^6)$

One may note that the asymptotic behavior of all of these solutions is locally AdS for  $\Lambda < 0$  provided  $|\Lambda| < 21/(80\alpha)$ , locally dS for  $\Lambda > 0$  and locally flat for  $\Lambda = 0$ .

Note that all the different  $F(r)$ 's given in this section have the same form as  $\alpha$  goes to zero, reducing to the solutions of Einstein gravity.

## 4.1 Taub-NUT Solutions

Using the conditions for NUT solutions, (i)  $F(r = N) = 0$  and (ii)  $F'(r = N) = 1/(4N)$ , we find that Gauss-Bonnet gravity in eight dimensions admits non-extreme NUT solutions only when the base space is chosen to be  $\mathbb{CP}^3$ . The conditions for a nonsingular NUT solution are satisfied provided the mass parameter is fixed to be

$$m_N = -\frac{8N^3}{105}(16\Lambda N^4 + 42N^2 - 105\alpha) \quad (28)$$

On the other hand, the solutions with  $\mathcal{B} = T^2 \times T^2 \times T^2 = \mathcal{B}_A$  and  $\mathcal{B} = T^2 \times T^2 \times S^2 = \mathcal{B}_B$  are external NUT solutions provided the mass parameter is

$$m_n^A = -\frac{128\Lambda N^7}{105}, \quad (29)$$

$$m_n^B = -\frac{16N^5}{105}(8\Lambda N^2 + 7) \quad (30)$$

These results for eight-dimensional Gauss-Bonnet gravity are consistent with our conjectures. Again, one may note that the former extremal NUT solution does not have a curvature singularity at  $r = N$  whereas the latter does.

## 4.2 Taub-Bolt Solutions

The conditions for having a regular bolt solution are (i)  $F(r = r_b) = 0$  and  $F'(r_b) = 1/(4N)$  with  $r_b > N$ . Condition (ii) again follows from the fact that we want to avoid a conical singularity at the bolt, together with the fact that the period of  $\tau$  will still be  $16\pi N$ . Now applying these conditions for  $\mathcal{B} = \mathbb{CP}^3$  gives the following equation for  $r_b$ :

$$4N\Lambda r_b^4 + 3r_b^3 - 4(3N + 2\Lambda N^3)r_b^2 + 3(8\alpha - N^2)r_b + 4N(\Lambda N^4 + 3N^2 - 9\alpha) = 0 \quad (31)$$

where again one has two bolt solutions for negative  $\Lambda$  provided  $N < N_{\max}$  where  $N_{\max}$  is the smaller root of the following equation

$$\begin{aligned} & -320\Lambda^2(27 + 64\Lambda\alpha)(15 + 64\Lambda\alpha)N^{10} - 2880\Lambda(9 + 16\Lambda\alpha)(15 + 64\Lambda\alpha)N^8 \\ & + 9(1179648\Lambda^3\alpha^3 + 110182\Lambda^2\alpha^2 + 185760\Lambda\alpha - 2025)N^6 \\ & - 216\alpha(43008\Lambda^2\alpha^2 + 8992\Lambda\alpha - 375)N^4 \\ & + 324\alpha^2(12288\Lambda^2\alpha^2 + 2688\Lambda\alpha - 301)N^2 + 41472\alpha^3 = 0 \end{aligned} \quad (32)$$

As  $\Lambda$  goes to zero Eq. (31) becomes

$$r_b^3 - 4Nr_b^2 + (8\alpha - N^2)r_b + 4N(N^2 - 3\alpha) = 0 \quad (33)$$

which holds for locally asymptotically flat solutions. For all  $N$ , there is only one solution for which  $r_b > N$ .

Taub-bolt solutions for the case of  $\mathcal{B} = S^2 \times \mathbb{CP}^2$  exist provided  $N \leq N_{\max}$  where  $N_{\max}$  is now given by the smallest real root of the following equation

$$\begin{aligned} & -4800\Lambda^2(243 + 896\Lambda\alpha)(27 + 128\Lambda\alpha)N^{10} - 2880\Lambda(499712\Lambda^2\alpha^2 + 1260496\Lambda\alpha + 32805)N^8 \\ & + (5532286976\Lambda^3\alpha^3 + 3243829248\Lambda^2\alpha^2 + 422703360\Lambda\alpha - 4428675)N^6 \\ & - 72\alpha(3381248\Lambda^2\alpha^2 + 623232\Lambda\alpha - 32265)N^4 \\ & + 139968\alpha^2(6912\Lambda^2\alpha^2 + 1472\Lambda\alpha - 159)N^2 + 10077696\alpha^3 = 0 \end{aligned} \quad (34)$$

In this case there exist two  $r_b$ 's which are the real roots of

$$4N\Lambda r_b^4 + 3r_b^3 - 4(3N + 2\Lambda N^3)r_b^2 + 3(8\alpha - N^2)r_b + 4N(\Lambda N^4 + 3N^2 - \frac{32}{3}\alpha) = 0 \quad (35)$$

Next we consider the Taub-bolt solutions for  $\mathcal{B} = S^2 \times S^2 \times S^2$ . One finds that  $r_b$  is given by the solution of the following equation

$$4N\Lambda r_b^4 + 3r_b^3 - 4(3N + 2\Lambda N^3)r_b^2 + 3(8\alpha - N^2)r_b + 4N(\Lambda N^4 + 3N^2 - 12\alpha) = 0 \quad (36)$$

One has two bolt solutions for  $N < N_{\max}$  and one solution for  $N = N_{\max}$  where  $N_{\max}$  is now given by the smallest real root of the following equation

$$\begin{aligned} & -64\Lambda^2(27 + 128\Lambda\alpha)(25 + 128\Lambda\alpha)N^{10} - 44\Lambda(45056\Lambda^2\alpha^2 + 19152\Lambda\alpha + 2025)N^8 \\ & + (12058624\Lambda^3\alpha^3 + 5548032\Lambda^2\alpha^2 + 597888\Lambda\alpha - 6075)N^6 \\ & - 72\alpha(38912\Lambda^2\alpha^2 + 6144\Lambda\alpha - 417)N^4 \\ & + 27648\alpha^2(48\Lambda^2\alpha^2 + 10\Lambda\alpha - 1)N^2 + 13824\alpha^3 = 0 \end{aligned} \quad (37)$$

For the locally asymptotic flat case,  $r_b$  is the solution to the equation

$$r_b^3 - 4Nr_b^2 + (8\alpha - N^2)r_b + 12N(N^2 - 4\alpha) = 0 \quad (38)$$

and using reasoning similar to the 6-dimensional case we have only one bolt solution.

For the case of  $\mathcal{B} = T^2 \times T^2 \times T^2$  and  $\mathcal{B} = T^2 \times T^2 \times S^2$ , Euclidean regularity at the bolt requires the period of  $\tau$  to be

$$\beta = -\frac{12\pi r_b}{\Lambda(r_b^2 - N^2)} \quad (39)$$

and

$$\beta = \frac{4\pi r_b(3r_b^2 - 3N^2 + 8\alpha)}{(r_b^2 - N^2)[1 - \Lambda(r_b^2 - N^2)]} \quad (40)$$

respectively. As  $r_b$  varies from  $N$  to infinity, one covers the whole temperature range from 0 to  $\infty$ , and therefore one can have bolt solutions. Again, one may note that for the case of asymptotic locally flat solution with base space  $\mathcal{B} = T^2 \times T^2 \times T^2$ , there is no black hole solution.

As with the six-dimensional case, there is a maximum value for  $\alpha$  for all base spaces that ensures all bolt and NUT solutions are regular in eight dimensions (as in the case of static solutions).

## 5 Ten-dimensional Solutions

In ten dimensions there are more possibilities for the base space  $\mathcal{B}$ . It can be an 8-dimensional space, the product of a 6-dimensional space with a 2-dimensional one, a product of two 4-dimensional spaces, a product of a 4-dimensional space with two 2-dimensional spaces, or the product of four 2-dimensional spaces. In all of these cases the form of the function  $F(r)$  is

$$\begin{aligned} F(r) &= \frac{(r^2 - N^2)^2}{12\alpha(7r^2 + 3N^2)} \left( 1 + \frac{3p\alpha}{2(r^2 - N^2)} - \sqrt{B(r) + C(r)} \right), \\ B(r) &= 1 + \frac{36\alpha m r (7r^2 + 3N^2)}{(r^2 - N^2)^6} + \frac{6p\alpha N^2}{35(r^2 - N^2)^6} (3r^8 - 28N^2 r^6 + 210N^4 r^4 + 420N^6 r^2 + 35N^8) \\ &\quad + \frac{2\alpha\Lambda(7r^2 + 3N^2)}{21(r^2 - N^2)^6} (7r^{10} - 45N^2 r^8 - 126N^4 r^6 - 210N^6 r^4 + 315N^8 r^2 + 63N^{10}) \end{aligned} \quad (41)$$

where  $p$  is the dimensionality of the curved portion of the base space, and the function  $C(r)$  depends on the choice of the base space  $\mathcal{B}$ .

The first case is when one chooses the 8-dimensional base space to be  $\mathbb{CP}^4$ . In this case the one-form  $A$  and the metric of  $\mathbb{CP}^4$  may be written as:

$$A_4 = 10 \sin^2 \xi_4 \left( d\psi_4 + \frac{1}{8} A_3 \right) \quad (42)$$

$$d\Sigma_3^2 = 10 \{ d\xi_4^2 + \sin^2 \xi_4 \cos^2 \xi_4 (d\psi_4 + \frac{1}{8} A_3)^2 + \frac{1}{8} \sin^2 \xi_4 d\Sigma_3^2 \} \quad (43)$$

where  $A_3$  and  $d\Sigma_3^2$  are given by Eqs. (27) and (26)

The 1-forms for all the other cases have been introduced in the previous sections, so we give only the function  $C(r)$  for different base spaces in the following table:

$\mathcal{B}$	$(r^2 - N^2)^6 C(r)/\alpha^2$
$\mathbb{CP}^4$	$-\frac{144}{25}(3r^8 - 28N^2r^6 + 210N^4r^4 + 420N^6r^2 + 35N^8)$
$T^2 \times T^2 \times T^2 \times T^2$	0
$T^2 \times T^2 \times T^2 \times S^2$	$9(r^2 - N^2)^4$
$T^2 \times T^2 \times S^2 \times S^2$	$\frac{12}{5}(r^8 + 4N^2r^6 - 90N^4r^4 - 220N^6r^2 - 15N^8)$
$T^2 \times S^2 \times S^2 \times S^2$	$-\frac{9}{5}(11r^8 - 76N^2r^6 + 450N^4r^4 + 820N^6r^2 + 75N^8)$
$S^2 \times S^2 \times S^2 \times S^2$	$-\frac{285}{5}(r^8 - 6N^2r^6 + 300N^4r^4 + 50N^6r^2 + 5N^8)$
$T^2 \times \mathbb{CP}^3$	$\frac{27}{5}(r^8 + 4N^2r^6 - 90N^4r^4 - 220N^6r^2 - 15N^8)$
$S^2 \times \mathbb{CP}^3$	$-\frac{18}{5}(9r^8 - 64N^2r^6 + 390N^4r^4 + 720N^6r^2 + 65N^8)$
$T^2 \times T^2 \times \mathbb{CP}^2$	$\frac{4}{5}(17r^8 - 52N^2r^6 - 90N^4r^4 - 500N^6r^2 - 15N^8)$
$T^2 \times S^2 \times \mathbb{CP}^2$	$-\frac{1}{5}(43r^8 - 428N^2r^6 + 3330N^4r^4 + 6740N^6r^2 + 555N^8)$
$S^2 \times S^2 \times \mathbb{CP}^2$	$-\frac{8}{5}(29r^8 - 184N^2r^6 + 990N^4r^4 + 1720N^6r^2 + 165N^8)$
$\mathbb{CP}^2 \times \mathbb{CP}^2$	$-\frac{16}{5}(11r^8 - 76N^2r^6 + 450N^4r^4 + 820N^6r^2 + 75N^8)$

Note that the asymptotic behavior of all of these solutions is locally AdS for  $\Lambda < 0$  provided  $|\Lambda| < 9/(42\alpha)$ , locally dS for  $\Lambda > 0$  and locally flat for  $\Lambda = 0$ . As with the 6 and 8 dimensional cases, all the different  $F(r)$ 's have the same form as  $\alpha$  goes to zero and reduce to the solutions of Einstein gravity.

## 5.1 Taub-NUT Solutions

Using the conditions for NUT solutions, (i)  $F(r = N) = 0$  and (ii)  $F'(r = N) = 1/(5N)$ , we find that Gauss-Bonnet gravity in ten dimensions admits non-extreme NUT solutions only when the base space is chosen to be  $\mathbb{CP}^4$ . There is no curvature singularity at  $r = N$  for this solution provided the mass parameter is fixed to be

$$m_n = -\frac{128N^5}{4725}(25\Lambda N^4 + 90N^2 - 378\alpha) \quad (44)$$

On the other hand, the solutions with  $\mathcal{B} = T^2 \times T^2 \times T^2 \times T^2 = \mathcal{B}_A$  and  $\mathcal{B} = T^2 \times T^2 \times T^2 \times S^2 = \mathcal{B}_B$  are external NUT solution provided the mass parameter is

$$m_n^A = -\frac{128\Lambda N^9}{189}, \quad (45)$$

$$m_n^B = -\frac{64N^7}{945}(10\Lambda N^2 + 9) \quad (46)$$

These results for ten-dimensional Gauss-Bonnet gravity are consistent with our conjectures – we have found that all other spaces with 2-dimensional factors in the base yield singular solutions. It is also straightforward to show that the former extremal NUT solution has no curvature singularity at  $r = N$ , whereas the latter has.

## 5.2 Taub-Bolt Solutions

The conditions for having a regular bolt solution are (i)  $F(r = r_b) = 0$  and  $F'(r_b) = 1/(5N)$  with  $r_b > N$ . Now applying these conditions for  $\mathcal{B} = \mathbb{CP}^4$  gives the following equation for

$r_b$ :

$$5N\Lambda r_b^4 + 4r_b^3 - 10N(2 + 2\Lambda N^2)r_b^2 + 4(12\alpha - N^2)r_b + N(5\Lambda N^4 + 20N^2 - 96\alpha) = 0 \quad (47)$$

where there exists a maximum value for the NUT parameter given by the smallest real root of the following equation

$$\begin{aligned} & -25\Lambda^2(4 + 15\Lambda\alpha)(1 + 15\Lambda\alpha)N^{10} - 50\Lambda(8 + 27\Lambda\alpha)(1 + 15\Lambda\alpha)N^8 \\ & + (34500\Lambda^3\alpha^3 + 18875\Lambda^2\alpha^2 + 2320\Lambda\alpha - 16)N^6 \\ & - 2\alpha(6975\Lambda^2\alpha^2 + 1205\Lambda\alpha - 48)N^4 \\ & + \alpha^2(6075\Lambda^2\alpha^2 + 1170\Lambda\alpha - 109)N^2 + 48\alpha^3 = 0 \end{aligned} \quad (48)$$

in order for bolt solutions to exist. For the locally asymptotic flat case,  $r_b$  is the solution of

$$r_b^3 - 5Nr_b^2 + (12\alpha - N^2)r_b + N(5N^2 - 24\alpha) = 0 \quad (49)$$

which yields only a single bolt solution.

For the case of  $\mathcal{B} = T^2 \times T^2 \times T^2 \times T^2$  and  $\mathcal{B} = S^2 \times T^2 \times T^2 \times T^2$ , Euclidean regularity at the bolt requires the period of  $\tau$  to be

$$\beta = -\frac{16\pi r_b}{\Lambda(r_b^2 - N^2)} \quad (50)$$

and

$$\beta = \frac{16\pi r_b(r_b^2 - N^2 + 3\alpha)}{(r_b^2 - N^2)[1 - \Lambda(r_b^2 - N^2)]} \quad (51)$$

respectively. As  $r_b$  varies from  $N$  to infinity, one covers the whole temperature range from 0 to  $\infty$ , and therefore one can have bolt solutions. Again, one may note that for the case of asymptotic locally flat solution with base space  $\mathcal{B} = T^2 \times T^2 \times T^2$ , there is no black hole solution.

Gauss-Bonnet gravity has other Taub-bolt solutions in ten dimensions corresponding to the various base space factors given in the above table. We shall omit the details of these solutions here, and simply note that it is sufficient to apply the regularity and reality conditions for bolt solutions mentioned at the beginning of this subsection to find the equation for  $r_b$ .

## 6 The General Form of the Solutions in $d$ Dimensions

In this section we write down the solutions of field equation (4) for the metric (5). We find that the form of  $F(r)$  in any dimension and for any choice of the base space is

$$F(r) = \frac{(r^2 - N^2)\{r^2 - N^2 + \frac{2(d-4)}{d-2}p\alpha\}}{2(d-4)\alpha\{(d-3)r^2 + 3N^2\}} - \sqrt{F_2(r)} \quad (52)$$

where  $p$  is the sum of the dimensions of the curved factor spaces of  $\mathcal{B}$ .

Here, we consider only the NUT/bolt solutions of those cases for which Gauss-Bonnet gravity admits NUT solutions, leaving out the other cases which have only bolt solution for reasons of economy. Our conjecture implies that there are three cases that have NUT solutions in  $2k + 2$  dimensions.

## 6.1 Taub-NUT/bolt solutions for the base space $\mathcal{B} = \mathbb{CP}^k$

The only case for which Gauss-Bonnet gravity admits non-extreme NUT solutions in  $2k+2$  dimensions is when the base space is  $\mathcal{B} = \mathbb{CP}^k$ . In this case the metric may be written as [24]

$$d\Sigma_k^2 = (2k+2) \left\{ d\xi_k^2 + \sin^2 \xi_k \cos^2 \xi_k (d\psi_k + \frac{1}{2k} A_{k-1})^2 + \frac{1}{2k} \sin^2 \xi_k d\Sigma_{k-1}^2 \right\} \quad (53)$$

where  $A_{k-1}$  is the Kähler potential of  $\mathbb{CP}^{k-1}$ . Here  $\xi_k$  and  $\psi_k$  are the extra coordinates corresponding to  $\mathbb{CP}^k$  with respect to  $\mathbb{CP}^{k-1}$ . Also, the metric is normalized such that, Ricci tensor is equal to the metric,  $R_{\mu\nu} = g_{\mu\nu}$ . The 1-form  $A_k$ , which is the Kähler potential of  $\mathbb{CP}^k$ , is

$$A_k = (2k+2) \sin^2 \xi_k (d\psi_k + \frac{1}{2k} A_{k-1}) \quad (54)$$

Now the  $tt$ -component of the field equation can be written as:

$$\Gamma_1 r F'(r) + \Gamma_2 F^2(r) + \Gamma_3 F(r) + \Gamma_4 = 0, \quad (55)$$

where  $\Gamma_1, \Gamma_2, \Gamma_3$  and  $\Gamma_4$  are

$$\begin{aligned} \Gamma_1 &= -\alpha \left( (d-3)r^2 + 3N^2 \right) F(r) + (r^2 - N^2) \left( \alpha + \frac{r^2 - N^2}{2(d-4)} \right), \\ \Gamma_2 &= -\frac{\alpha}{2(r^2 - N^2)} \left\{ (d-3)(d-5)r^4 + 2(d-9)N^2 r^2 + 3N^4 \right\}, \\ \Gamma_3 &= -\frac{(r^2 - N^2)[(d-3)r^2 + N^2]}{2(d-4)} + \alpha \left( (d-5)r^2 + N^2 \right), \\ \Gamma_4 &= (r^2 - N^2) \left\{ \Lambda \frac{(r^2 - N^2)^2}{(d-2)(d-4)} - \frac{(r^2 - N^2)}{2(d-4)} - \frac{(d-2)\alpha}{2d} \right\} \end{aligned} \quad (56)$$

We can write the general form of  $F_2(r)$  in  $d$  dimensions. Inserting the general form of  $F(r)$  given in Eq. (52) into the differential equation (55), we find  $F_2(r)$  for  $\mathcal{B} = \mathbb{CP}^k$  as:

$$F_2(r) = \frac{r \left( (-1)^{(d-2)/2} m + \frac{1}{d(d-2)(d-4)^2 \alpha} \int \Upsilon(r) dr \right)}{4\alpha \{ (d-3)r^2 + 3N^2 \} (r^2 - N^2)^{(d-6)/2}} \quad (57)$$

where  $\Upsilon(r)$  is

$$\begin{aligned} \Upsilon(r) &= \frac{(r^2 - N^2)^{d/2}}{r^2 [(d-3)r^2 + 3N^2]^2} \left\{ -24(d-2)(d-3)(d-4)^2 \alpha^2 + 8d(d-2)\alpha \Lambda [(d-3)r^2 + 3N^2]^2 \right. \\ &\quad \left. + 24d(d-2)(d-4)\alpha N^2 + d(d-2)[(d-1)(d-3)r^4 + 6(d-1)r^2 N^2 + 3N^4] \right\} \end{aligned} \quad (58)$$

The solutions given by Eqs. (52) and (57) yield a NUT solution for any given (even) dimension  $d > 4$  provided the mass parameter  $m$  is fixed to be

$$m_{\text{nut}} = -\frac{(\frac{d}{2} - 3)! 2^{d/2} N^{d-5}}{2d(d-1)!!} \left\{ 2d^2 \Lambda N^4 + d(d-1)(d-2)N^2 - (d-1)(d-2)(d-3)(d-4)\alpha \right\} \quad (59)$$

This solution has no curvature singularity at  $r = N$ .

Solutions given by Eqs. (52) and (57) for  $m \neq m_{\text{nut}}$  in any dimension can be regarded as bolt solutions. The value of the bolt radius  $r_b > N$  may be found from the regularity conditions (i)  $F(r = r_b) = 0$  and  $F'(r_b) = 2/(dN)$ . Applying these for  $\mathcal{B} = \mathbb{CP}^k$  gives the following equation for  $r_b$ :

$$0 = 2d\Lambda N r_b^4 + 2(d-2)r_b^3 - dN [(d-2) + 4\Lambda N^2] r_b^2 + 2(d-2) [2(d-4)\alpha - N^2] r_b + N [2d\Lambda N^4 + d(d-2)N^2 - (d-4)(d-2)^2\alpha] \quad (60)$$

which has two real roots larger than  $N$ , provided  $N < N_{\text{max}}$  and one for  $N = N_{\text{max}}$ , where  $N_{\text{max}}$  is the smaller real root of the following equation

$$\begin{aligned} & -4(d+2)d^2[8d(d-4)\alpha\Lambda + d^2 - 4][8d(d-4)(d-6)\alpha\Lambda + d^3 - 6d^2 + 12d - 8]\Lambda^2 N^{10} \\ & -2d^2(d-2)(d+2)[8d(d-4)\alpha\Lambda + d^2 - 4][8(d-4)(d^2 - 6d - 4)\alpha\Lambda + d^3 - 6d^2 + 12d - 8]\Lambda N^8 \\ & + (d-2)^2\{128d^3(d-4)^3(d^2 - 4d - 14)\alpha^3\Lambda^3 + 32d^2(d-4)^2(d^4 - 4d^3 - 4d^2 + 40d + 40)\Lambda^2\alpha^2 \\ & + 2d(d-4)(d^2 - 4)(d^4 - 4d^3 + 6d^2 + 80d + 24)\alpha\Lambda - d^6 + 4d^5 + 4d^4 - 32d^3 + 16d^2 + 64d - 64\}N^6 \\ & - (d-4)(d-2)^2\{192d^2(d-4)^2(d^2 + 2d + 4)\alpha^2\Lambda^2 + 16d(d-2)(d-4)(d^3 + 8d^2 + 14d - 12)\alpha\Lambda \\ & - d^6 + 6d^5 - 16d^4 - 48d^3 + 176d^2 + 96d - 384\}\alpha N^4 + (d-4)^2(d-2)^2\{1728d^2(d-4)^2\alpha^2\Lambda^2 \\ & + 48d(d-2)(5d^2 - 18d - 8)\alpha\Lambda - 13d^4 + 16d^3 + 8d^2 + 192d - 336\}\alpha^2 N^2 \\ & + 128(d-2)^4(d-4)^3\alpha^3 = 0 \end{aligned} \quad (61)$$

For the locally asymptotically flat case,  $r_b$  is the solution of

$$2r_b^3 - dN r_b^2 + 2 [2(d-4)\alpha - N^2] r_b + N [dN^2 - (d-4)(d-2)\alpha] = 0 \quad (62)$$

where there is no condition in order for a bolt solution to exist, except in six dimensions for which there exists a lower limit of the NUT charge.

Next we write down the general form of the solutions with the base space  $\mathcal{B} = T^2 \times \dots \times T^2$ . The field equation is given by (55), where now

$$\begin{aligned} \Gamma_1 &= -\alpha\{(d-3)r^2 + 3N^2\}F + \frac{(r^2 - N^2)^2}{2(d-4)}, \\ \Gamma_2 &= -\frac{\alpha}{2(r^2 - N^2)} \{(d-3)(d-5)r^4 + 2(d-9)N^2r^2 + 3N^4\}, \\ \Gamma_3 &= -\frac{(r^2 - N^2)[(d-3)r^2 + N^2]}{2(d-4)}, \\ \Gamma_4 &= \Lambda \frac{(r^2 - N^2)^3}{(d-2)(d-4)} \end{aligned} \quad (63)$$

Inserting the general form of  $F(r)$  given in Eq. (52) into the differential equation (55) with  $\Gamma_i$ 's of Eqs. (63), we find that  $F_2(r)$  for  $\mathcal{B} = T^2 \times T^2 \times \dots \times T^2$  is

$$F_2(r) = \frac{r \left( (-1)^{(d-2)/2} m + \frac{1}{(d-2)(d-4)^2\alpha} \int \Upsilon(r) dr \right)}{4\alpha\{(d-3)r^2 + 3N^2\}(r^2 - N^2)^{(d-6)/2}} \quad (64)$$

where  $\Upsilon(r)$  is

$$\Upsilon(r) = \frac{(r^2 - N^2)^{d/2}}{r^2 \{(d-3)r^2 + 3N^2\}^2} \{8(d-4)\alpha\Lambda[(d-3)r^2 + 3N^2]^2 + (d-2)[(d-1)(d-3)r^4 + 6(d-1)r^2N^2 + 3N^4]\}$$

The solutions given by Eqs. (52) and (64) yield a NUT solution for any given (even) dimension  $d > 4$  provided the mass parameter  $m$  is fixed to be

$$m_{\text{nut}} = -\frac{d(\frac{d}{2} - 3)!2^{d/2}}{(d-1)!!} \Lambda N^{d-1} \quad (65)$$

where in this case the spacetime has no curvature singularity at  $r = N$ . Also one may find that Euclidean regularity at the bolt requires the period of  $\tau$  to be

$$\beta = -\frac{2(d-2)\pi r_b}{\Lambda(r_b^2 - N^2)} \quad (66)$$

and can have any value from zero to infinity as  $r_b$  varies from  $N$  to infinity, and therefore one can have bolt solution.

Finally, we consider the solution when  $\mathcal{B} = S^2 \times T^2 \times \dots \times T^2$ . In this case the field has the same form as Eq. (55) with

$$\begin{aligned} \Gamma_1 &= -\alpha\{(d-3)r^2 + 3N^2\}F + (r^2 - N^2)\left\{\frac{2}{d-2}\alpha + \frac{r^2 - N^2}{2(d-4)}\right\}, \\ \Gamma_2 &= -\frac{\alpha}{2(r^2 - N^2)} \left\{(d-3)(d-5)r^4 + 2(d-9)N^2r^2 + 3N^4\right\}, \\ \Gamma_3 &= -\frac{(r^2 - N^2)[(d-3)r^2 + N^2]}{2(d-4)} + \frac{4(d-4)}{d-2}\alpha\{(d-5)r^2 + N^2\}, \\ \Gamma_4 &= \frac{(r^2 - N^2)^2[\Lambda(r^2 - N^2) - 1]}{(d-2)(d-4)} \end{aligned} \quad (67)$$

Inserting the general form of  $F(r)$  given in Eq. (52) into the differential equation (55) with  $\Gamma_i$ 's of Eqs. (67), we find that  $F_2(r)$  for  $\mathcal{B} = T^2 \times T^2 \times \dots \times T^2$  is

$$F_2(r) = \frac{r \left( (-1)^{(d-2)/2} m + \frac{1}{(d-2)^2(d-4)^2\alpha} \int \Upsilon(r) dr \right)}{4\alpha\{(d-3)r^2 + 3N^2\}(r^2 - N^2)^{(d-6)/2}} \quad (68)$$

where  $\Upsilon(r)$  is

$$\begin{aligned} \Upsilon(r) &= \frac{(r^2 - N^2)^{d/2}}{r^2 \{(d-3)r^2 + 3N^2\}^2} \left\{ 16(d-4)^2\alpha^2 \frac{(d-3)(d-5)r^4 + 6(d-3)N^2r^2 + 3N^4}{(r^2 - N^2)^2} \right. \\ &\quad + 8(d-2)(d-4)\alpha\Lambda[(d-3)r^2 + 3N^2]^2 + 48(d-2)(d-4)\alpha N^2 \\ &\quad \left. - (d-2)^2[(d-1)(d-3)r^4 + 6(d-1)r^2N^2 + 3N^4] \right\} \end{aligned} \quad (69)$$



The solutions given by Eqs. (52) and (68) yield a NUT solution for any given (even) dimension  $d \geq 4$  with curvature singularity at  $r = N$ , provided the mass parameter  $m$  is fixed to be

$$m_{\text{nut}} = -\frac{(\frac{d}{2} - 3)!2^{d/2}}{(d-1)!!}N^{d-3} \{d\Lambda N^2 + (d-1)\} \quad (70)$$

Also one may find that the Euclidean regularity at the bolt requires the period of  $\tau$  to be

$$\beta = \frac{2(d-2)\pi r_b(r_b^2 - N^2 + \frac{4(d-4)}{d-2}\alpha)}{(r_b^2 - N^2)[1 - \Lambda(r_b^2 - N^2)]} \quad (71)$$

Again,  $\beta$  of Eq. (71) can have any value from zero to infinity as  $r_b$  varies from  $N$  to infinity, and therefore one can have bolt solution.

The asymptotic behavior of all of these solutions is locally AdS for  $\Lambda < 0$  provided  $|\Lambda| < (d-1)(d-2)/[(d-3)(d-4)\alpha]$  locally dS for  $\Lambda > 0$  and locally flat for  $\Lambda = 0$ . All the different  $F(r)$ 's for differing base spaces have the same form as  $\alpha$  goes to zero and reduce to the solutions of Einstein gravity.

## 7 Concluding Remarks

We have considered the existence of Taub-NUT/bolt solutions in Gauss-Bonnet gravity with and without cosmological term. These solutions are constructed as circle fibrations over even dimensional spaces that in general are products of Einstein-Kähler spaces. We found that the function  $F(r)$  of the metric depends on the specific form of the base factors on which one constructs the circle fibration. In other words we found that the solutions are sensitive to the geometry of the base space, in contrast to Einstein gravity where the metric in any dimension is independent of the specific form of the base factors. We restricted ourselves to the cases of base spaces with zero or positive curvature factor spaces, since when the base space has 2-hyperboloids with negative curvature, then the function  $F(r)$  is not real for the whole range of  $r$  when  $\alpha$  is positive.

We found that when Einstein gravity admits non-extremal NUT solutions with no curvature singularity at  $r = N$ , then there exists a non-extremal NUT solution in Gauss-Bonnet gravity. In  $(2k+2)$ -dimensional spacetime, this happens when the metric of the base space is chosen to be  $\mathbb{CP}^k$ . Indeed, Gauss-Bonnet gravity does not admit non-extreme NUT solutions with any other base space. We also found that when the base space has at most a 2-dimensional curved factor space with positive curvature, then Gauss-Bonnet gravity admits extremal NUT solutions where the temperature of the horizon at  $r = N$  vanishes. We have extended these observations to two conjectures about the existence of NUT solutions in Gauss-Bonnet gravity.

We also found the bolt solutions of Gauss-Bonnet gravity in various dimensions and different base spaces, and gave the equations which can be solved for the horizon radius of the bolt solution. For  $\Lambda \neq 0$ , there is a maximal value for the NUT charge in terms of the parameters of the metric that  $N$  must be less than or equal to that in order to have bolt solutions. In this case for  $\Lambda < 0$  we have two bolt solutions in Gauss-Bonnet gravity if  $N < N_{\text{max}}$  and one bolt solution if  $N = N_{\text{max}}$ . The value of  $N_{\text{max}}$  is larger in Gauss-Bonnet

gravity than in Einstein gravity. For the case of  $\Lambda = 0$  we always have only one bolt solution. In the 6-dimensional case with the base space  $\mathbb{CP}^2$  the bolt solution exists provided the NUT charge is larger than its minimal value of  $\sqrt{\alpha}$ . This situation is quite unlike that in Einstein gravity, where the NUT charge can smoothly approach zero. We note that Gauss-Bonnet gravity in six dimensions gives the most general second order differential equation in classical gravity, while in higher dimensions in order to have the most general second order differential equation, one should turn on other higher terms of Lovelock gravity.

Our solutions have been generalized in an obvious way for even dimensions higher than ten. We gave the differential equation and the general form of the function  $F(r)$  in any arbitrary even dimensions. For instance, in twelve dimensions, the base space is ten-dimensional and in general can be factorized as a 10-dimensional space, the product of an 8-dimensional space with a 2-dimensional one, a product of 6-dimensional space with a 4-dimensional space, a product of a 6-dimensional space with two 2-dimensional spaces, a product of two 4-dimensional spaces with a 2-dimensional space, a product of a 4-dimensional spaces with three 2-dimensional spaces, or the product of five 2-dimensional spaces. In any dimension  $2k + 2$ , we have only one non-extremal NUT solution with  $\mathbb{CP}^k$  as the base space, and two extremal NUT solutions with the base spaces  $T^2 \times T^2 \times \dots \times T^2$  and  $S^2 \times T^2 \times T^2 \times \dots \times T^2$ . There is no curvature singularity for the first two case, while for the latter case, the spacetime has curvature singularity at  $r = N$ .

Insofar as Gauss-Bonnet gravity is expected to model leading-order quantum corrections to Einstein gravity, we see that quantum effects can be expected to single out a preferred non-singular base space and to yield a minimal value for the NUT charge in six dimensions. Thermodynamically this corresponds to a maximal temperature. The study of thermodynamic properties of these solutions, the investigation of the existence of NUT solutions in continued Lovelock gravity, or Lovelock gravity with higher order terms remain to be carried out in future.

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